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New Simple Method for Obtaining Integrable Hierarchies of Soliton Equations with Multicomponent Potential Functions*

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A new simple method for obtaining integrable hierarchies of soliton equations is proposed. First of all, a new loop algebra $\tilde{G}_M$ is constructed, whose commutation operation is clear as that in loop algebra $\tilde{A}_1$. Second, by making use of the Tu scheme, many of integrable hierarchies with multicomponent potential functions can be produced. As a specific application of our method, a multicomponent AKNS hierarchy is obtained. Finally, an expanding loop algebra $\tilde{F}_M$ of the loop algebra $\tilde{G}$ is constructed. Taking advantage of $\tilde{F}_M$ above, a type of integrable coupling system of the multicomponent AKNS hierarchy is worked out.

KEY WORDS: loop algebra; multicomponent integrable hierarchy; integrable coupling; Tu scheme.

1. INTRODUCTION

Search for new integrable hierarchies has been an important and interesting topic in soliton theory. One used various methods to have obtained integrable Hamiltonian hierarchies possessing physical senses (Gu, 1990; Li, 1999; Li and Zhuang, 1982; Wadati, Konno, and Ichikawa, 1979; Shimizu and Wadati, 1980; Tu, 1989; Ma, 1992, 1993a,b,c; Hu, 1994, 1997; Fan, 2000, 2002). As far as integrable hierarchies with multipotential functions are concerned, Guo Fukui obtained a few interesting results in Guo (2000, 2002). Recently, professors Ma Wenxiu and Zhou Ruguang used generalized Tu scheme to work out multicomponent AKNS hierarchy in (Ma and Zhou, 2002). In this paper, we propose a simple new method to generate integrable hierarchies of soliton equations with multicomponent potential functions. As an illustrative example, we deduce a multicomponent ANKS

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hierarchy by constructing a type of new loop algebra $\tilde{G}_M$ and employing Tu scheme. Furthermore, an expanding loop algebra $\tilde{F}_M$ of the loop algebra $\tilde{G}_M$ is constructed. It follows that a kind of integrable coupling of the multicomponent AKNS hierarchy is engendered by use of the theory of integrable couplings (Fuchssteiner, 1993; Ma and Fuchssteiner, 1996). The method proposed in this paper can be used generally.

2. A NEW LOOP ALGEBRA

Set

$$G_M = \{a = (a_{a_{ij}})_{M\times 3} = (a_1, a_2, a_3)\} \tag{1}$$

where M stands for an arbitrary positive integer, a_{ij} are real or complex numbers, a_i $(i = 1, 2, 3)$ are columns of the vector a.

Then G_M is a linear space.

Definition 2.1. Set two vectors to be $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^{\mathrm{T}}$, $\beta = (\beta_1, \beta_2, \ldots, \beta_M)^{\mathrm{T}}$, and define a vector product $\alpha * \beta$ as

$$\alpha * \beta = \beta * \alpha = (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_M\beta_M)^{\mathrm{T}} \tag{2}$$

Introducing the diagonal matrix $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_M)$, obviously, we have

$$\alpha * \beta = \tilde{\alpha}\beta \tag{3}$$

Definition 2.2. Set $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$ to be two elements in G_M, define a commutation operation $[a, b]$ as

$$[a, b] = (a_2 * b_3 - a_3 * b_2, a_1 * b_2 - a_2 * b_1, a_3 * b_1 - a_1 * b_3) \tag{4}$$

It is easy to verify that the operation (4) is linear and antisymmetric. Furthermore, a direct calculation reads that for $\forall a, b, c \in G_M$,

$$[[a, b], c] + [[b, c], a] + [[c, a,], b] = 0 \tag{5}$$

which signifies Jacobian identity holds. Therefore, G_M along with the operation (4) becomes a Lie algebra.

Denoting

$$\tilde{G}_M = \{a\lambda^n, a \in G_M, n = 0, \pm 1, \pm 2, \ldots\} \tag{6}$$

with a commutation operation.

$$[a\lambda^m, b\lambda^n] = [a, b]\lambda^{m+n}, \quad a, b \in G_M \tag{7}$$

Obviously, $\tilde{G}_M$ is a loop algebra.

Consider a Lax pair as follows:

$$\begin{cases}\phi_x = [U, \phi], \quad \lambda_t = 0, \phi, U, V, \in \tilde{G}_M \\ \phi_t = [V, \phi]\end{cases} \tag{8}$$

whose compatibility gives rise to the following:

$$\begin{aligned}&\phi_{xt}[U_t, \phi] + [U, [V, \phi]] = \phi_{tx} = [V_x, \phi] + [V, [U, \phi]] \\ &[U_t, \phi] - [V_x, \phi] + [U, [V, \phi]] + [V, [\phi, U]] = 0\end{aligned} \tag{9}$$

With the help of (5), (9) can be rewritten as

$$[U_t, \phi] - [V_x, \phi] + [[U, V], \phi] = 0 \tag{10}$$

Due to ϕ being arbitrary, (10) reduces to

$$U_t - V_x + [U, V] = 0 \tag{11}$$

i.e., the zero-curvature equation holds, from which multicomponent integrable hierarchies are Lax integrable.

In what follows, we omit the symbol $*$ for operation convenience.

3. A MULTICOMPONENT AKNS HIERARCHY AND ITS INTEGRABLE COUPLING

Consider an isospectral problem

$$\phi_x = U\phi, \quad U = (\lambda I_M, q, r) \tag{12}$$

where

$$I_M = \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}_{M\times 1}, \quad q = \begin{pmatrix}q_1\\q_2,\\\vdots\\q_M\end{pmatrix}, \quad r = \begin{pmatrix}r_1\\r_2\\\vdots\\r_M\end{pmatrix}$$

Set

$$V = \sum_{m\geq 0}(a(0,m) + \lambda a(1,m), b(0,m) + \lambda b(1,m), c(0,m) + \lambda c(1,m))\lambda^{-2m}$$

where

$$a(0,m) = \begin{pmatrix} a_{m1}^{(0)} \\ a_{m2}^{(0)} \\ \vdots \\ a_{mM}^{(0)} \end{pmatrix}, \quad a(1,m) = \begin{pmatrix} a_{m1}^{(1)} \\ a_{m2}^{(1)} \\ \vdots \\ a_{mM}^{(1)} \end{pmatrix}, \ldots$$

Solving the stationary zero curvature equation

$$V_x = [U, V] \tag{13}$$

leads to

$$\begin{cases} a_x(0,m) = qc(0,m) - rb(0,m), a_x(1,m) = qc(1,m) - rb(1,m), \\ b(1,m+1) = b_x(0,m) + qa(0,m), b_x(1,m) = b(0,m) - qa(1,m), \\ c(1,m+1) = ra(0,m) - c_x(0,m), c_x(1,m) + ra(1,m) - c(0,m), \\ a(0,0) = \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^{\mathrm{T}}, \alpha_i \text{ are constants} \\ b(0,0) = c(0,0) = 0, a(1,1) = 0, b(1,1) = \alpha q, c(1,1) = \alpha r, \\ a(0,1) = -\alpha qr, b(0,1) = \alpha q_x, c(0,1) = -\alpha r_x \end{cases} \tag{14}$$

Note

$$\begin{cases} V_+^{(n)} = \sum\limits_{m=0}^{n} (a(0,m) + \lambda a(1,m), b(0,m) + \lambda b(1,m), c(0,m) + \lambda c(1,m))\lambda^{2n-2m} \\ V_-^{(n)} = \lambda^{2n} V - V_+^{(n)} \end{cases}$$

then (13) is turned into

$$-V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] = V_{-x}^{(n)} - \left[U, V_-^{(n)}\right] \tag{15}$$

A direct calculation gives

$$-V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] = (0, -b(1,n+1), c(1,n+1))$$

Denoting $V^{(n)} = V_+^{(n)}$, the zero curvature equation

$$U_t - V_{+x}^{(n)} + \left[U, V^{(n)}\right] = 0 \tag{16}$$

admits the Lax integrable system

$$u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -b(1,n+1) \\ c(1,n+1) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c(1,n+1) \\ b(1,n+1) \end{pmatrix} = J \begin{pmatrix} c(1,n+1) \\ b(1,n+1) \end{pmatrix} \tag{17}$$

where J is a symplectic operator.

From (14), we have

$$\begin{pmatrix} c(1,n+1) \\ b(1,n+1) \end{pmatrix} = \begin{pmatrix} r\partial^{-1}q - \partial & -r\partial^{-1}r \\ q\partial^{-1}q & \partial - q\partial^{-1}r \end{pmatrix} \begin{pmatrix} c(0,n) \\ b(0,n) \end{pmatrix} = L \begin{pmatrix} c(0,n) \\ b(0,n) \end{pmatrix}$$
$$= L^2 \begin{pmatrix} c(1,n) \\ b(1,n) \end{pmatrix}$$

Hence, the system (17) can be written as

$$u_t \begin{pmatrix} q \\ r \end{pmatrix}_t = JL \begin{pmatrix} c(0,n) \\ b(0,n) \end{pmatrix} = JL^{2n} \begin{pmatrix} \alpha r \\ \alpha q \end{pmatrix} \tag{18}$$

which is a multicomponent AKNS hierarchy we look for.

To search for integrable couplings of the hierarchy (18), we first construct an expanding Lie algebra of the Lie algebra G_M as follows:

$$F_M = \{a = (a_{ij})_{M\times 5} = (a_1, a_2, \ldots, a_5)\} \tag{19}$$

Definition 3.3. For arbitrary vectors $a = (a_1, a_2, \ldots, a_5)$, $b = (b_1, b_2, \ldots, b_5) \in F_M$, define a commutation operation as

$$[a, b] = (a_2b_3 - a_3b_2, a_1b_2 - a_2b_1, a_3b_1 - a_1b_3, a_1b_4 - a_4b_1 + a_2b_5 - a_5b_2,$$
$$a_3b_4 - a_4b_3 + a_5b_1 - a_1b_5) \tag{20}$$

It is easy to find that F_M with the operation (20) consists of a Lie algebra.

Again set

$$\tilde{F}_M = \{a\lambda^n, a \in F_M, n = 0, \pm 1, \ldots\} \tag{21}$$

then $\tilde{F}_M$ is an expanding loop algebra of the loop algebra $\tilde{G}_M$ since taking $a_4 = a_5 = 0$, (21) reduces to $\tilde{G}_M$.

Taking two subalgebras $\tilde{F}_M(1)$ and $\tilde{F}_M(2)$ as

$$\tilde{F}_M(1) = \{(a_1, a_2, a_3, 0, 0)\lambda^n\}, \quad \tilde{F}_M(2) = \{(0, 0, 0, a_4, a_5)\lambda^n\}$$

then we find

(i) $\tilde{F}_M = \tilde{F}_M(1) \oplus \tilde{F}_M(2)$, $\tilde{F}_M(1) \cong \tilde{G}_M$,
(ii) $[\tilde{F}_M(1), \tilde{F}_M(2)] \subset \tilde{F}_M(2)$,

where the symbol $\cong$ stands for isomorphic relations.

By employing the relations (i) and (ii), we can work out a type of integrable coupling of the multicomponent AKNS hierarchy.

Consider an isospectral problem

$$\phi_x = U\phi, \quad U = (\lambda I_M, q, r, u_1, u_2) \tag{22}$$

let $V = \sum_{m\geq 0}(a(0,m)+\lambda a(1,m), b(0,m)+\lambda b(1,m), c(0,m)+\lambda c(1,m), \lambda d(0, m), \lambda d(1,m), f(0,m)+\lambda f(1,m))\lambda^{-2m}$, solving an adjoint equation similar to (13) yields

$$\begin{cases} a_x(0,m) = qc(0,m) - rb(0,m), a_x(1,m) = qc(1,m) - rb(1,m), \\ b(1,m+1) = b_x(0,m) + qa(0,m), b_x(1,m) = b(0,m) - qa(1,m), \\ c(1,m+1) = ra(0,m) - c_x(0,m), c_x(1,m) = ra(1,m) - c(0,m), \\ d_x(0,m) = -u_1a(0,m) + qf(0,m), -u_2b(0,m) + d(1,m+1), \\ d_x(1,m) = d(0,m) - u_1a(1,m), -u_2b(1,m) + qf(1,m), \\ f_x(0,m) = rd(0,m) - u_1c(0,m) + u_2a(0,m) - f(1,m+1), \\ f_x(1,m) = rd(1,m) - u_1c(1,m) + u_2a(1,m) - f(0,m), \\ a(0,0) = \alpha = (\alpha_1, \alpha_2, \ldots \alpha_M)^{\mathrm{T}}, b(0,0) = c(0,0) = d(0,0) = f(0,0) \\ \quad = a(1,1) = 0, \\ b(1,1) = \alpha q, c(1,1) = \alpha r, a(0,1) = -\alpha qr, b(0,1) = \alpha q_x, c(0,1) = -\alpha r_x, \\ d(1,1) = \alpha_{U_1}, f(1,1) = \alpha_{U_2}, f(0,1) = -\alpha u_{2x}, d(0,1) = \alpha u_{1x} \end{cases} \tag{23}$$

Similar to (15), we have

$$\begin{aligned} -V_{+x}^{(n)} + \left[U, V_+^{(n)}\right] &= (0, -b(1,n+1), c(1,n+1), -d(1,n+1) \\ &+ u_1a(1,n+1) + u_2b(1,n+1), u_1c(1,n+1), -u_2a(1,n+1) + f(1,n+1)) \end{aligned}$$

Taking $V^{(n)} = V_+^{(n)}$, solving the zero curvature equation similar to (16) produces

$$\begin{aligned} u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix} &= \begin{pmatrix} -b(1,n+1) \\ c(1,n+1) \\ d(1,n+1) - u_1a(1,n+1) - u_2b(1,n+1) \\ -f(1,n+1) - u_1c(1,n+1) + u_2a(1,n+1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -u_1\partial^{-1}q & u_1\partial^{-1}r - u_2 & 0 & 1 \\ -u_1 + u_2\partial^{-1}q & -u_2\partial^{-1}r & -1 & 0 \end{pmatrix} \begin{pmatrix} c(1,n+1) \\ b(1,n+1) \\ f(1,n+1) \\ d(1,n+1) \end{pmatrix} \\ &= J \begin{pmatrix} c(1,n+1) \\ b(1,n+1) \\ f(1,n+1) \\ d(1,n+1) \end{pmatrix} \end{aligned} \tag{24}$$

From (23), a recurrence operator L meets

$$\begin{pmatrix} c(1,n+1) \\ b(1,n+1) \\ f(1,n+1) \\ d(1,n+1) \end{pmatrix} = \begin{pmatrix} r\partial^{-1}q-\partial & -r\partial^{-1}r & 0 & 0 \\ q\partial^{-1}q & \partial-q\partial^{-1}r & 0 & 0 \\ -u_1+u_2\partial^{-1}q & -u_2\partial^{-1}r & -\partial & r \\ u_1\partial^{-1}q & u_2-u_1\partial^{-1}r & -q & \partial \end{pmatrix} \begin{pmatrix} c(0,n) \\ b(0,n) \\ f(0,n) \\ d(0,n) \end{pmatrix}$$

$$= L \begin{pmatrix} c(0,n) \\ b(0,n) \\ f(0,n) \\ d(0,n) \end{pmatrix} = L^2 \begin{pmatrix} c(1,n) \\ b(1,n) \\ f(1,n) \\ d(1,n) \end{pmatrix}$$

Hence, the system (24) can be written as

$$u_t = \begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix} = JL^{2n} \begin{pmatrix} \alpha r \\ \alpha q \\ \alpha u_2 \\ \alpha u_1 \end{pmatrix} \tag{25}$$

According to the definition of integrable couplings, we conclude that the system (25) is a type of integrable coupling of the system (18). Taking $u_1 = u_2 = 0$, the system (25) reduces to (18). Therefore the hierarchy (25) is also a type of expanding integrable model of the system (18).

Remark 3.4. We proposed a simple method for generating integrable hierarchies of soliton equations with multicomponent potential functions. Especially, we obtained the multicomponent AKNS hierarchy and its multicomponent integrable coupling. However, there is an open problem. The results obtained by our method is Lax integrable. How do we obtain Liouville integrable multicomponent hierarchies and their Hamiltonian structures, conserved laws by improving our method? This is worth studying in the future.

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